Electromagnetic Waves (Light) — Chapter 30

PHYS 2203 — Introductory Physics III

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Overview

Starting from Maxwell's equations, we will derive the existence of electromagnetic waves. We will see that the speed of light emerges not as an empirical input but as a prediction of electromagnetism. Along the way we will meet the wave equation and verify its standing-wave solution.

1 Maxwell's Equations

The four equations that govern classical electricity and magnetism are stated together below.

$$\textbf{Gauss's Law:} \qquad \oint_S E_n \, dA = \frac{Q_{\text{inside}}}{\varepsilon_0} \tag{1}$$

$$\textbf{Gauss's Law for Magnetism:} \qquad \oint_S B_n \, dA = 0 \tag{2}$$

$$\textbf{Faraday's Law:} \qquad \oint_C \vec{E} \cdot d\vec{\ell} = -\int_S \frac{\partial B_n}{\partial t} \, dA \tag{3}$$

$$\textbf{Ampère's Law:} \qquad \oint_C \vec{B} \cdot d\vec{\ell} = \mu_0 I + \mu_0 \varepsilon_0 \int_S \frac{\partial E_n}{\partial t} \, dA \tag{4}$$

The symbols:

$\vec{E}$: electric field	$\vec{B}$: magnetic field
E_n : $\vec{E}$ component normal to dA	B_n : $\vec{B}$ component normal to dA
ε_0 : permittivity of free space	μ_0 : permeability of free space
Q_{inside} : enclosed charge	I : current
S : surface	C : path

The surface integrals in the last two equations are the **electric flux** and **magnetic flux** through S,

$$\Phi_E \equiv \int_S E_n \, dA, \qquad \Phi_B \equiv \int_S B_n \, dA, \tag{5}$$

so for a stationary loop Faraday's and Ampère's laws can be written compactly as

$$\oint_C \vec{E} \cdot d\vec{\ell} = -\frac{d\Phi_B}{dt}, \qquad \oint_C \vec{B} \cdot d\vec{\ell} = \mu_0 I + \mu_0 \varepsilon_0 \frac{d\Phi_E}{dt}. \tag{6}$$

These flux forms are the ones we will apply to infinitesimal loops below.

Key observation: Faraday's law states that a time-varying $\vec{B}$ produces a circulating $\vec{E}$, and Ampère's law (with Maxwell's displacement-current term) states that a time-varying $\vec{E}$ produces a circulating $\vec{B}$. The mutual coupling of these two effects is the origin of electromagnetic wave propagation.

2 Parallel Plate Capacitor: Setting Up the Wave

A parallel plate capacitor driven by a sinusoidal voltage $V \sim V_0 \sin \omega t$ provides a region of free space with a time-varying electric field. As the voltage oscillates, so do the plate charges and hence the field: a uniform $\vec{E}_0$ along $\hat{y}$ between the plates, and a fringing field $\vec{E}_{\text{fringe}}$ bowing outward at the edges. In the surrounding vacuum, $\partial \vec{E}/\partial t \neq 0$.

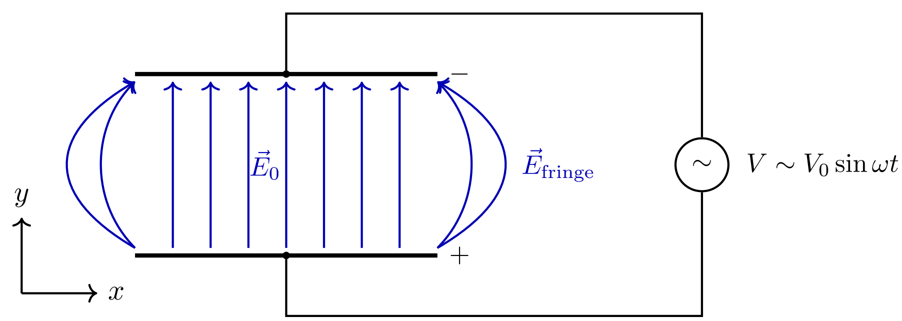

Figure 1: Parallel plate capacitor driven by a sinusoidal voltage source. Uniform $\vec{E}_0$ between the plates, fringing fields $\vec{E}_{\text{fringe}}$ at the edges.

Our goal: find the $\vec{B}$ field that accompanies this time-varying $\vec{E}$, and show that the two propagate together as a wave.

Plane-wave simplification

Away from the capacitor we track a disturbance propagating along $+\hat{x}$ and assume the simplest field configuration consistent with it: each field has a single component, depending only on x and t,

$$\vec{E} = E_y(x,t)\,\hat{y}, \qquad \vec{B} = B_z(x,t)\,\hat{z}. \tag{7}$$

We now apply Faraday's and Ampère's laws to two infinitesimal loops at a representative point — one in the x–y plane, one in the x–z plane. Each yields one PDE, and the two PDEs combine into a wave equation.

3 Applying Faraday's Law (Loop in the x–y Plane)

Apply Faraday's law, $\oint \vec{E}\cdot d\vec{\ell} = -d\Phi_B/dt$, to an infinitesimal rectangle in the x–y plane at $(x,0,0)$: width dx along $\hat{x}$, height ℓ along $\hat{y}$, traversed counter-clockwise starting at the top-left corner ($\bullet$).

Fields on this loop:

- On the *left* edge (at x) the field is $E_y(x)$. On the *right* edge (at $x+dx$) it has grown to $E_y(x+dx) = E_y + dE_y$.

- The magnetic field B_z threads the loop, out of the page.

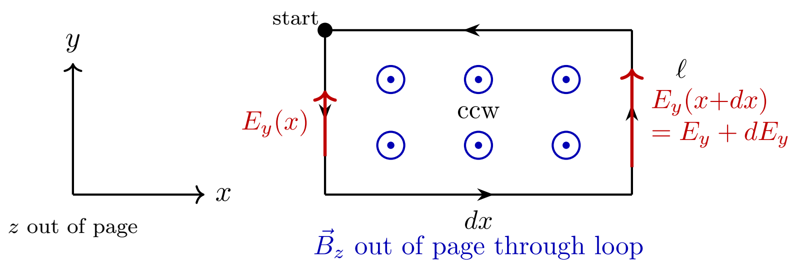

Figure 2: Infinitesimal Faraday loop in the x–y plane at $(x, 0, 0)$. The red arrows show the field E_y on the two vertical edges (slightly larger on the right since E_y varies with x). The blue $\odot$ symbols show B_z pointing out of the page through the loop. The path is traversed counter-clockwise starting at the top-left corner: $-\hat{y}$ (left edge), $+\hat{x}$ (bottom), $+\hat{y}$ (right edge), $-\hat{x}$ (top).

Walk around the loop segment by segment, keeping track of the direction of travel on each edge. Since $\vec{E}$ has only a y-component, only the two vertical edges contribute:

1. **Top-left → bottom-left** (left edge, $-\hat{y}$, length ℓ).
   $\vec{E} = E_y(x)\,\hat{y}$, $\quad d\vec{\ell} = -\hat{y}\,d\ell$. $\quad$ **Term:** $-E_y\ell$**.**

2. **Bottom-left → bottom-right** ($+\hat{x}$, length dx).
   $E_x = 0$. $\quad$ **Term:** 0**.**

3. **Bottom-right → top-right** (right edge, $+\hat{y}$, length ℓ).
   $\vec{E} = (E_y + dE_y)\,\hat{y}$, $\quad d\vec{\ell} = +\hat{y}\,d\ell$. $\quad$ **Term:** $+(E_y + dE_y)\ell$**.**

4. **Top-right → top-left** ($-\hat{x}$, length dx).
   $E_x = 0$. $\quad$ **Term:** 0**.**

The $\pm E_y\ell$ pieces cancel:

$$\oint \vec{E} \cdot d\vec{\ell} \;=\; -E_y\ell + (E_y + dE_y)\ell \;=\; dE_y\,\ell. \tag{8}$$

Right-hand side — the flux. A counter-clockwise path has normal $+\hat{z}$ (right-hand rule), so $\Phi_B = B_z\,\ell\,dx$, and Faraday's law reads

$$dE_y\,\ell \;=\; -\frac{d}{dt}\big(B_z\,\ell\,dx\big). \tag{9}$$

Promoting to partial derivatives. Since $E_y(x, t)$ depends on two variables, its total differential is $dE_y = \frac{\partial E_y}{\partial x}\,dx + \frac{\partial E_y}{\partial t}\,dt$. Here dE_y compares the two edges at one instant, so $dt = 0$ and $dE_y = (\partial E_y/\partial x)\,dx$. The loop is stationary, so the flux changes only through $\partial B_z/\partial t$. Dividing by $\ell\,dx$:

$$\boxed{\frac{\partial E_y}{\partial x} = -\frac{\partial B_z}{\partial t}} \tag{10}$$

Physical interpretation: a spatially varying E_y must be accompanied by a time-varying B_z. This is the first of the two equations we need.

4 Applying Ampère's Law (Loop in the x–z Plane)

Faraday gave one PDE relating E_y and B_z; Ampère's law gives the second. No charge flows through the vacuum, so $I = 0$ and only the displacement-current term contributes.

Apply Ampère's law to an infinitesimal rectangle in the x–z plane at $(x, 0, 0)$: width dx along $\hat{x}$, height ℓ along $\hat{z}$. **Orientation matters here:** with x to the right and z up in the figure, right-handed axes put $+\hat{y}$ *into* the page, so E_y threads the loop as $\otimes$. We choose the loop normal along $+\hat{y}$ (parallel to $\vec{E}$). The right-hand rule then fixes the traversal shown, starting at the bottom-left corner ($\bullet$) and going up the left edge first.

Fields on this loop:

- On the *left* edge (at x) the magnetic field is $B_z(x)$. On the *right* edge (at $x + dx$) it is $B_z(x + dx) = B_z + dB_z$.
- The electric field E_y threads the loop, into the page ($\otimes$).

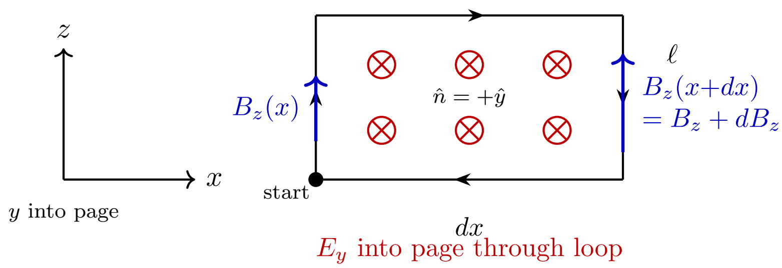

Figure 3: Infinitesimal Ampère loop in the x–z plane at $(x, 0, 0)$. The blue arrows show B_z on the two vertical edges (larger on the right since B_z varies with x). The red $\otimes$ symbols show E_y threading the loop along $+\hat{y}$, which points *into* the page for right-handed axes drawn with x right and z up. The traversal (start at bottom-left: $+\hat{z}$ up the left edge, $+\hat{x}$ along the top, $-\hat{z}$ down the right edge, $-\hat{x}$ along the bottom) is right-handed about the normal $\hat{n} = +\hat{y}$.

Walk around the loop segment by segment. Since $\vec{B}$ has only a z-component, only the two vertical edges contribute:

1. **Bottom-left → top-left** (left edge, $+\hat{z}$, length ℓ).
   $\vec{B} = B_z(x)\,\hat{z}$, $\quad d\vec{\ell} = +\hat{z}\,d\ell$. $\quad$ **Term:** $+B_z\ell$.
2. **Top-left → top-right** ($+\hat{x}$, length dx).
   $B_x = 0$. $\quad$ **Term:** 0.
3. **Top-right → bottom-right** (right edge, $-\hat{z}$, length ℓ).
   $\vec{B} = (B_z + dB_z)\,\hat{z}$, $\quad d\vec{\ell} = -\hat{z}\,d\ell$. $\quad$ **Term:** $-(B_z + dB_z)\ell$.
4. **Bottom-right → bottom-left** ($-\hat{x}$, length dx).
   $B_x = 0$. $\quad$ **Term:** 0.

The $\pm B_z\ell$ pieces cancel:

$$\oint \vec{B} \cdot d\vec{\ell} \;=\; B_z\ell - (B_z + dB_z)\ell \;=\; -dB_z\,\ell. \tag{11}$$

Right-hand side — the displacement current. With $I = 0$ and normal $+\hat{y}$, the electric flux is $\Phi_E = E_y\,\ell\,dx$, and Ampère's law reads

$$-dB_z\,\ell \;=\; \mu_0\varepsilon_0\,\frac{d}{dt}\big(E_y\,\ell\,dx\big). \tag{12}$$

Promoting to partial derivatives as before ($dB_z = (\partial B_z/\partial x)\,dx$ at fixed time) and dividing by $\ell\,dx$:

$$\boxed{-\frac{\partial B_z}{\partial x} \;=\; \mu_0\varepsilon_0\,\frac{\partial E_y}{\partial t}} \tag{13}$$

Physical interpretation: a spatially varying B_z must be accompanied by a time-varying E_y — the counterpart of the Faraday result. We now have two equations relating E_y and B_z, so we can eliminate one field and solve for the other.

5 Deriving the Wave Equation

We have two coupled first-order PDEs:

$$\frac{\partial E_y}{\partial x} = -\frac{\partial B_z}{\partial t} \qquad \text{and} \qquad -\frac{\partial B_z}{\partial x} = \mu_0\varepsilon_0\,\frac{\partial E_y}{\partial t}.$$

Strategy: differentiate each once more — with respect to x and with respect to t — then eliminate one field by substitution, leaving a single second-order PDE for each field alone.

$$① \quad \frac{\partial E_y}{\partial x} = -\frac{\partial B_z}{\partial t} \implies \frac{\partial^2 E_y}{\partial x^2} = -\frac{\partial^2 B_z}{\partial x\,\partial t} \text{ and } \frac{\partial^2 E_y}{\partial x\,\partial t} = -\frac{\partial^2 B_z}{\partial t^2} \quad ② \tag{14}$$

$$③ \quad -\frac{\partial B_z}{\partial x} = \mu_0\varepsilon_0\frac{\partial E_y}{\partial t} \implies -\frac{\partial^2 B_z}{\partial x\,\partial t} = \mu_0\varepsilon_0\frac{\partial^2 E_y}{\partial t^2} \text{ and } -\frac{\partial^2 B_z}{\partial x^2} = \mu_0\varepsilon_0\frac{\partial^2 E_y}{\partial t\,\partial x} \quad ④ \tag{15}$$

Combine ① and ③ (the mixed partials $\partial^2 B_z/\partial x\,\partial t$ are equal):

$$\boxed{\frac{\partial^2 E_y}{\partial t^2} = \frac{1}{\mu_0\varepsilon_0}\,\frac{\partial^2 E_y}{\partial x^2}} \tag{16}$$

Combine ② and ④:

$$\boxed{\frac{\partial^2 B_z}{\partial t^2} = \frac{1}{\mu_0\varepsilon_0}\,\frac{\partial^2 B_z}{\partial x^2}} \tag{17}$$

These are **wave equations** with speed

$$\boxed{c = \frac{1}{\sqrt{\mu_0\varepsilon_0}} \approx 2.998\times 10^8\ \text{m/s} \quad \text{“speed of light”}} \tag{18}$$

The constants ε_0 and μ_0 are measured in tabletop experiments — ε_0 from Coulomb's law, μ_0 from forces between current-carrying wires — with no light involved anywhere. Yet $1/\sqrt{\mu_0\varepsilon_0}$ reproduces the measured speed of light. This result, first obtained by Maxwell in 1865, identifies light as an electromagnetic wave.

6 Wave Equation in Other Physical Systems

The same equation,

$$\frac{\partial^2 f(x,t)}{\partial t^2} = v^2 \frac{\partial^2 f(x,t)}{\partial x^2}, \tag{19}$$

describes many physical systems:

Example 1 — Guitar string: $f(x,t)$ = transverse displacement of stretched string,

$$v = \sqrt{\frac{\text{string tension}}{\text{mass/length of string}}} \tag{20}$$

Example 2 — Sound waves in air: $f(x,t)$ = air pressure,

$$v = \text{speed of sound in air } (\approx 340 \text{ m/s}) \tag{21}$$

Because the equation is identical, so are its solutions: whatever we learn about $f(x,t)$ applies directly to $E_y(x,t)$ and $B_z(x,t)$.

7 Solutions to the Wave Equation

What functions solve the wave equation? Since it is second order in both x and t, we expect a four-parameter family of sinusoidal solutions. Two physically distinct forms are useful: the **standing wave** (fixed spatial profile, amplitude oscillating in time) and the **traveling wave** (profile moving rigidly at speed v). We treat the standing wave here and verify it by direct substitution. We will cover the traveling wave in the next lecture.

Standing Wave Solution

$$\boxed{f(x,t) = A\sin(kx+\theta)\cos(\omega t+\phi)} \qquad \text{where } v = \frac{\omega}{k} \tag{22}$$

The four constants: A (amplitude), k (wave number), θ and ϕ (phases); ω is fixed by $\omega = vk$.

Verify by direct substitution:

$$\frac{\partial^2 f(x,t)}{\partial t^2} = -\omega^2 f(x,t) \tag{23}$$

$$\frac{\partial^2 f(x,t)}{\partial x^2} = -k^2 f(x,t) \tag{24}$$

Plugging into the wave equation:

$$-\omega^2 f(x,t) = v^2\big(-k^2 f(x,t)\big) \implies v^2 = \frac{\omega^2}{k^2} \quad \checkmark \tag{25}$$

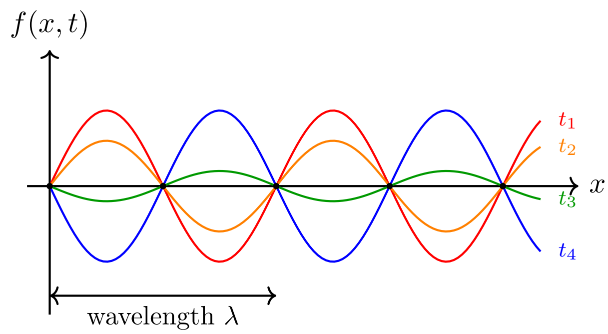

Figure 4: Standing wave snapshots at four times $t_1 < t_2 < t_3 < t_4$. Nodes (dots) are stationary. The envelope oscillates. The period of the example is $T = 2(t_4 - t_1)$.

Key quantities:

$$\text{wave number:} \quad k = \frac{2\pi}{\lambda} \qquad [\mathrm{m}^{-1}] \tag{26}$$

$$\text{angular frequency:} \quad \omega = \frac{2\pi}{T} \qquad [\mathrm{s}^{-1}] \tag{27}$$

$$\text{frequency:} \quad f = \frac{1}{T} \qquad [\mathrm{Hz}] \tag{28}$$

Once the medium fixes v, wavelength and period are not independent: $v = \omega/k = \lambda/T = f\lambda$.