Energy and Momentum of Electromagnetic Waves

PHYS 2203 — Introductory Physics III

August 26, 2026

Overview

This lecture completes the free-space description of electromagnetic waves and begins to study the mechanical properties of light. We first finish the wave picture from the previous lecture: the traveling-wave solution and the relation $E_0 = cB_0$. We then ask how much energy and momentum an electromagnetic wave carries, and how it transports energy from place to place.

Recap: The Wave Equation for $\vec{E}$ and $\vec{B}$

Last lecture we applied Faraday's and Ampère's laws to a plane wave in vacuum and obtained two coupled first-order equations,

$$\frac{\partial E_y}{\partial x} = -\frac{\partial B_z}{\partial t}, \qquad -\frac{\partial B_z}{\partial x} = \mu_0 \varepsilon_0 \frac{\partial E_y}{\partial t}.$$

Cross-differentiating and eliminating one field decoupled these into a **wave equation** for each field separately,

$$\boxed{\frac{\partial^2 E_y}{\partial t^2} = c^2 \frac{\partial^2 E_y}{\partial x^2}} \qquad \text{and} \qquad \boxed{\frac{\partial^2 B_z}{\partial t^2} = c^2 \frac{\partial^2 B_z}{\partial x^2}},$$

both propagating at the same speed

$$c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} \approx 2.998 \times 10^8 \text{ m/s}.$$

We also wrote down and verified the **standing-wave** solution, $f(x,t) = A\sin(kx+\theta)\cos(\omega t + \phi)$. We now complete the picture: the **traveling-wave** solution, its consequences for the relation between $\vec{E}$ and $\vec{B}$, and the electromagnetic spectrum.

1 Traveling Wave Solution

A traveling wave is a disturbance that keeps its shape while moving along the x-axis. We construct one by letting x and t enter only through the combination $kx - \omega t$:

$$\boxed{f(x,t) = A\sin(kx - \omega t + \delta)} \tag{1}$$

Verify by direct substitution:

$$\frac{\partial^2 f(x,t)}{\partial t^2} = -\omega^2 f(x,t) \tag{2}$$

$$\frac{\partial^2 f(x,t)}{\partial x^2} = -k^2 f(x,t) \tag{3}$$

Plugging into the wave equation $\partial^2 f/\partial t^2 = v^2\, \partial^2 f/\partial x^2$:

$$-\omega^2 f = v^2\big(-k^2 f\big) \quad \Longrightarrow \quad \omega^2 = v^2 k^2 \quad \Longrightarrow \quad v = \frac{\omega}{k}. \tag{4}$$

So the form is a solution provided $\omega = vk$.

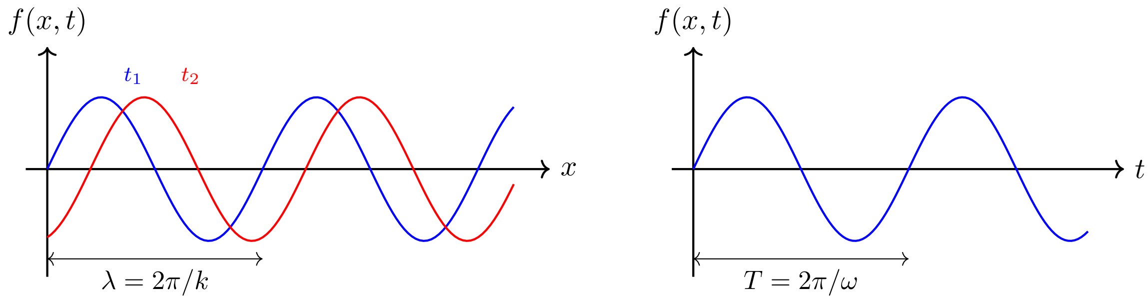
Speed of the wave. To see that this profile actually moves, and how fast, follow a point of constant f (a particular crest, for example). Its phase stays fixed as time advances:

$$kx - \omega t + \delta = \text{const.} \tag{5}$$

Differentiating with respect to t gives

$$k\,\frac{dx}{dt} - \omega = 0 \quad \Longrightarrow \quad \frac{dx}{dt} = \frac{\omega}{k} = v. \tag{6}$$

The crest, and with it the entire waveform, moves in the $+x$ direction at speed v.



(a) constant t (b) constant x

Figure 1: (a) Two snapshots of a traveling wave at t_1 and $t_2 > t_1$. The waveform shifts to the right by $v(t_2 - t_1)$. (b) The same wave observed at constant x vs. t, with period $T = 2\pi/\omega$.

Conventions:

- Usually choose $t = 0$ such that $\delta = 0$.
- $f(x,t) = A\sin(kx - \omega t)$ describes a wave traveling in the $+x$ direction with speed $v = \omega/k$, where A is the amplitude.
- $f(x,t) = A\sin(kx + \omega t)$ describes a wave traveling in the $-x$ direction with speed $v = \omega/k$.

2 Application to EM Waves

Each wave equation can be solved on its own, but E_y and B_z are not independent — they must also satisfy the original first-order equations (Faraday and Ampère). As we now show, this fixes the relative phase and the ratio of the amplitudes.

The wave equations for E_y and B_z in vacuum are:

$$\frac{\partial^2 E_y}{\partial t^2} = c^2 \frac{\partial^2 E_y}{\partial x^2} \qquad \text{and} \qquad \frac{\partial^2 B_z}{\partial t^2} = c^2 \frac{\partial^2 B_z}{\partial x^2} \tag{7}$$

Take traveling wave solutions:

$$E_y(x,t) = E_0 \sin(kx - \omega t + \delta), \qquad B_z(x,t) = B_0 \sin\left(kx - \omega t + \tilde{\delta}\right) \tag{8}$$

Recall Faraday's Law requires $\dfrac{\partial E_y}{\partial x} = -\dfrac{\partial B_z}{\partial t}$:

$$kE_0 \cos(kx - \omega t + \delta) = +\omega B_0 \cos\left(kx - \omega t + \tilde{\delta}\right) \qquad \text{for all } x \text{ and } t. \tag{9}$$

This forces $\delta = \tilde{\delta}$ (usually set $\delta = \tilde{\delta} = 0$) **and**

$$E_0 = \frac{\omega}{k} B_0 \implies \boxed{E_0 = cB_0} \tag{10}$$

Therefore:

$$\boxed{E_y(x,t) = E_0 \sin(kx - \omega t)} \qquad \boxed{B_z(x,t) = B_0 \sin(kx - \omega t)} \tag{11}$$

with $c = \dfrac{\omega}{k} = f\lambda$.

Three principal features of this result:

1. **Common phase.** $\vec{E}$ and $\vec{B}$ reach their maxima together, vanish together, and reverse together. They are in phase.

2. **Perpendicular geometry.** $\vec{E}$ lies along $\hat{y}$, $\vec{B}$ lies along $\hat{z}$, and propagation is along $\hat{x}$. The three directions are mutually perpendicular, identifying the wave as transverse. Electromagnetic waves in vacuum are always transverse.

3. **Fixed amplitude ratio.** $E_0 = cB_0$. For a charge q moving at speed v, the magnetic force is at most $qvB_0 = (v/c)\,qE_0$ — smaller than the electric force by a factor v/c. Since $v \ll c$ for ordinary charges, the force exerted by a light wave on a charged particle is dominated by the electric field.

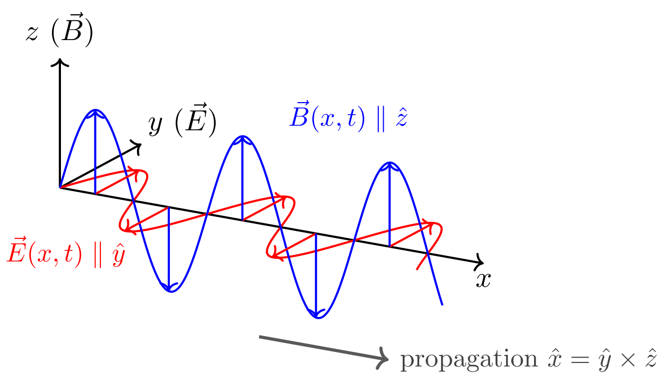


Figure 2: Plane EM wave traveling in $+\hat{x}$: $\vec{E}$ oscillates along $\hat{y}$, $\vec{B}$ along $\hat{z}$, in phase, with $E_0 = cB_0$. The triad $(\hat{x}, \hat{y}, \hat{z})$ is right-handed and the direction of propagation equals $\hat{E} \times \hat{B} = \hat{y} \times \hat{z} = \hat{x}$.

With the free-space wave now fully characterized, we turn to the energy and momentum it carries.

3 Energy Transport: The Poynting Vector and Intensity

Recall that in any region containing electric and magnetic fields, the local energy density (energy per unit volume) is the sum of an electrostatic contribution (Ch. 24) and a magnetostatic contribution (Ch. 28):

$$u \;=\; \frac{1}{2}\,\varepsilon_0 E^2 \;+\; \frac{B^2}{2\mu_0}. \tag{12}$$

Both terms have units of J/m^3.

Energy density of a plane wave

For the plane wave,

$$\vec{E} = E_0\,\sin(kx-\omega t)\,\hat{y}, \qquad \vec{B} = B_0\,\sin(kx-\omega t)\,\hat{z},$$

the amplitude relation $B = E/c$ makes the two contributions *equal*:

$$\frac{B^2}{2\mu_0} \;=\; \frac{E^2}{2\mu_0 c^2} \;=\; \frac{1}{2}\,\varepsilon_0 E^2, \tag{13}$$

using $c^2 = 1/(\mu_0\varepsilon_0)$. The electric and magnetic fields carry equal shares of the wave's energy, and the total energy density is

$$u \;=\; \varepsilon_0 E^2 \;=\; \varepsilon_0 E_0^2\,\sin^2(kx-\omega t). \tag{14}$$

Energy transport

Consider a fixed surface of area A perpendicular to the propagation direction. In a time dt the wave advances a distance $c\,dt$, so the energy that crosses the surface is all the energy in the slab of volume $A\,(c\,dt)$ just behind it:

$$dU \;=\; u\,(A\,c\,dt). \tag{15}$$

Power is energy delivered per unit time (dU/dt), so the power crossing the surface per unit area is

$$\frac{\text{power}}{\text{area}} \;=\; \frac{1}{A}\frac{dU}{dt} \;=\; u\,c. \tag{16}$$

Rewriting $u\,c$ in terms of the fields, again with $E = cB$:

$$u\,c \;=\; \varepsilon_0 c\,E^2 \;=\; \varepsilon_0 c^2\,EB \;=\; \frac{EB}{\mu_0}. \tag{17}$$

This combination is the magnitude of the **Poynting vector**

$$\boxed{\vec{S} \;=\; \frac{1}{\mu_0}\,\vec{E}\times\vec{B},} \tag{18}$$

whose direction $\hat{E}\times\hat{B} = \hat{x}$ is the direction of energy flow. For our plane wave,

$$\vec{S}(x,t) \;=\; \frac{1}{\mu_0}\,E_0 B_0\,\sin^2(kx-\omega t)\,\hat{x}, \tag{19}$$

the instantaneous power per unit area transported by the wave, flowing along the propagation direction. (The formula $\vec{S} = \vec{E}\times\vec{B}/\mu_0$ in fact gives the energy flux for *any* field configuration — a general result of Maxwell's equations known as Poynting's theorem.)

Intensity

The magnitude $|\vec{S}|$ oscillates between zero and E_0B_0/μ_0 twice per cycle — about 10^{15} times per second for visible light. No detector responds that fast: photodiodes, thermometers, and eyes all integrate the delivered energy over times far longer than an optical period. What a detector reports is therefore the *time-averaged* power per unit area at its fixed location, called the **intensity**:

$$I \;\equiv\; \left\langle \frac{\text{power}}{\text{area}} \right\rangle_t \;=\; \langle |\vec{S}| \rangle_t, \tag{20}$$

the energy delivered per unit area per unit time.

The **time average** of a quantity $g(t)$ over one period $T = 2\pi/\omega$ is its integral over a full cycle divided by the length of that cycle:

$$\langle g \rangle_t \;\equiv\; \frac{1}{T}\int_0^T g(t)\,dt. \tag{21}$$

We will need the average of $\sin^2$. Using the identity $\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$ and $T = 2\pi/\omega$,

$$\begin{aligned}
\langle \sin^2(kx-\omega t)\rangle_t &= \frac{1}{T}\int_0^T \tfrac{1}{2}\big[\,1-\cos(2kx-2\omega t)\,\big]\,dt && (22)\\
&= \frac{1}{2T}\left[t+\frac{1}{2\omega}\sin(2kx-2\omega t)\right]_0^T && (23)\\
&= \frac{1}{2T}\left[T+\frac{\sin(2kx-4\pi)-\sin(2kx)}{2\omega}\right] = \frac{1}{2}, && (24)
\end{aligned}$$

where $\sin(2kx-4\pi) = \sin(2kx)$ makes the last term vanish. Applying this to the Poynting magnitude gives the intensity:

$$I \;=\; \langle |\vec{S}| \rangle_t \;=\; \frac{1}{\mu_0}\,E_0B_0\,\langle \sin^2(kx-\omega t)\rangle_t \;=\; \frac{1}{2\mu_0}\,E_0B_0. \tag{25}$$

With $B_0 = E_0/c$ and $1/(c\mu_0) = c\varepsilon_0$, the intensity may be written entirely in terms of the electric field amplitude:

$$\boxed{I \;=\; \frac{1}{2\mu_0 c}\,E_0^2 \;=\; \frac{1}{2}\,c\varepsilon_0\,E_0^2.} \tag{26}$$

Either form has units of $\mathrm{W/m^2}$.

One immediate consequence: for a *point source* that radiates isotropically with total power L (called the **luminosity**), energy conservation requires the same power to cross every sphere centered on the source. A sphere of radius r has area $4\pi r^2$, so the intensity at distance r is

$$\boxed{I(r) \;=\; \frac{L}{4\pi r^2}} \tag{27}$$

— the **inverse-square law** for radiation.

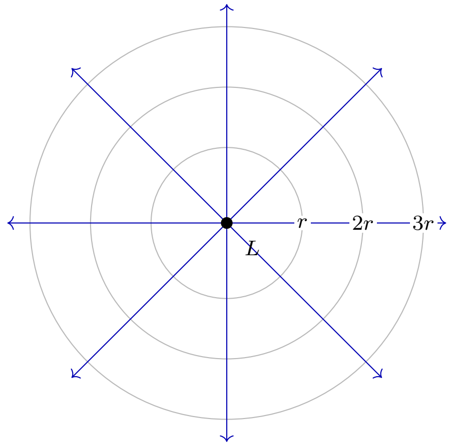


Figure 3: A point source of luminosity L radiating isotropically. The same total power crosses every sphere, so the intensity $I = L/4\pi r^2$ falls off as $1/r^2$.

4 Momentum Transport

Electromagnetic waves carry momentum as well as energy. To see why, follow a single charge q (mass m, initially at rest) struck by a plane wave with $\vec{E} = E\,\hat{y}$ and $\vec{B} = B\,\hat{z}$, treating the field magnitudes as constant over a short interval.

The electric field exerts a force qE on the charge along $\hat{y}$. By Newton's second law, $F = ma$, this produces an acceleration

$$a_y \;=\; \frac{qE}{m}. \tag{28}$$

The charge starts from rest, so after a time t its velocity is $v_y = a_y\,t = (qE/m)\,t$. By time t_1 it has gained the kinetic energy

$$U \;=\; \tfrac{1}{2}mv_y^2 \;=\; \tfrac{1}{2}\,m\left(\frac{qE}{m}\,t_1\right)^2 \;=\; \frac{1}{2}\,\frac{q^2E^2}{m}\,t_1^2. \tag{29}$$

Now that the charge is moving, it feels the magnetic force $q\vec{v}\times\vec{B} = qv_yB\,\hat{x}$, which points along the propagation direction and does no work (it is perpendicular to $\vec{v}$). Recall Newton's second law in its general form, $F = dp/dt$: force is the rate of change of momentum. Rearranging, $dp = F\,dt$, so the total momentum delivered by a force acting over a time interval is its integral,

$$p_x \;=\; \int_0^{t_1} F_x\,dt \;=\; \int_0^{t_1} qv_yB\,dt. \tag{30}$$

Substituting $v_y = (qE/m)\,t$ and integrating,

$$p_x \;=\; \frac{q^2EB}{m}\int_0^{t_1} t\,dt \;=\; \frac{1}{2}\,\frac{q^2EB}{m}\,t_1^2. \tag{31}$$

In the ratio p_x/U the charge, mass, and exposure time all cancel, leaving only $B/E = 1/c$:

$$\boxed{p \;=\; \frac{U}{c}} \tag{32}$$

Nothing about the charge survived in the ratio, so this is a property of the light wave itself: it carries momentum U/c even though it has no mass.